Announcements

1) Problems, notes added, due date 4/17

2) Online class?

3) Extra credit - Spivaks Partitions of Unity - fill in the blanks!

Definition: (k-tensor)

A k-tensor on $\mathbb{R}^n$ is simply a multilinear map from

$$\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}} \text{ to } \mathbb{R}$$

Notation: $\mathcal{T}^k(\mathbb{R}^n)$

Observe: $\mathcal{T}^k(\mathbb{R}^n)$ is

a vector space over $\mathbb{R}$

with the following operations:

let $T, S \in \mathcal{T}^k(\mathbb{R}^n)$, $c \in \mathbb{R}$

- $(T+S)(v_1, v_2, \ldots, v_k)$

$$= T(v_1, v_2, \ldots, v_k) + S(v_1, v_2, \ldots, v_k)$$

- $(cT)(v_1, v_2, \ldots, v_k)$

$$= c \cdot (T(v_1, v_2, \ldots, v_k))$$

$\forall \; v_1, v_2, \ldots, v_k \in \mathbb{R}^n.$

Definition: (tensor product)

Given $T \in \mathcal{T}^k(\mathbb{R}^n)$ and $S \in \mathcal{T}^m(\mathbb{R}^n)$, we define

$S \otimes T \in \mathcal{T}^{(k+m)}(\mathbb{R}^n)$ by

$$\boxed{(S \otimes T)(v_1, v_2, \dots, v_{k+m}) = S(v_1, v_2, \dots, v_m) \cdot T(v_{m+1}, v_{m+2}, \dots, v_{m+k})}$$

$\in \mathbb{R}$, the tensor product of S by T.

Properties: Let $S_1, S_2 \in \mathcal{T}^k(\mathbb{R}^n)$, $T_1, T_2 \in \mathcal{T}^m(\mathbb{R}^n)$. Then

1) $(S_1 + S_2) \otimes T_1$
$= (S_1 \otimes T_1) + (S_2 \otimes T_1)$

2) $S_1 \otimes (T_1 + T_2)$
$= (S_1 \otimes T_1) + (S_1 \otimes T_2)$

3) If $c \in \mathbb{R}$,

$$c(S_1 \otimes T_1)$$

$$= (cS_1) \otimes T_1$$

$$= S_1 \otimes (cT_1)$$

4) If $R_1 \in \mathcal{T}^\ell(\mathbb{R}^n)$,

$$(R_1 \otimes S_1) \otimes T_1 = R_1 \otimes (S_1 \otimes T_1)$$

However, it is not true in general that

$$S \otimes T = T \otimes S.$$

Example: $\varphi_1, \varphi_2 : \mathbb{R}^2 \to \mathbb{R}$

as defined before.

$$(\varphi_1 \otimes \varphi_2)(e_1, e_2)$$

$$= \varphi_1(e_1) \cdot \varphi_2(e_2) = 1, \text{ but}$$

$$(\varphi_2 \otimes \varphi_1)(e_1, e_2) = \varphi_2(e_1)\varphi_1(e_2)$$

$$= 0$$

Theorem: The dimension of $\mathcal{T}^k(\mathbb{R}^n)$ is n^k.

A basis is given by

$$\{\underbrace{\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}}_{k \text{ times}}\}_{i_j \in \{1,2,\dots,n\}}$$

$\forall 1 \le j \le k$

Proof:

Spanning: Let $T \in \mathcal{T}^k(\mathbb{R}^n)$.

Then

$$T = \sum_{i_1, i_2, \dots, i_k \in \{1,2,\dots,n\}} T(e_{i_1}, e_{i_2}, \dots, e_{i_k}) \left(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k} \right)$$

Only need to check on basis vectors and there the equality is immediate.

Linear Independence

Suppose $\exists$ scalars $c_{i_1, i_2, \ldots, i_k}$

for all $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$

with

$$\sum_{i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}} c_{i_1, i_2, \ldots, i_k} \left(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k} \right) = 0$$

where "0" is the multilinear map sending all vectors to $0 \in \mathbb{R}$.

Plug in basis elements again to obtain

$$c_{i_1, i_2, \dots, i_k} = 0$$

$$\forall \quad i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}. \qquad \square$$

Example 1: (inner product)

Recall $(\cdot,\cdot): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$(v,w) = \sum_{k=1}^{n} v_k w_k$$

if $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n)$.

write $(\cdot,\cdot) = \sum_{k=1}^{n} (\varphi_k \otimes \varphi_k)$

Example 2: (determinant)

Recall

$$\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$$

$$\det = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(\varphi_{\sigma(1)} \otimes \varphi_{\sigma(2)} \otimes \cdots \otimes \varphi_{\sigma(n)} \right)$$

2x2 case

$$\det\left(\begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix}, \begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}\right)$$

$$= v_{1,1} v_{2,2} - v_{1,2} v_{2,1}$$

$$= \varphi_1\left(\begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix}\right) \cdot \varphi_2\left(\begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}\right)$$

$$- \varphi_2\left(\begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix}\right) \cdot \varphi_1\left(\begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}\right)$$

$$= \varphi_1 \otimes \varphi_2 \left(\begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix}, \begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}\right)$$

$$- \varphi_2 \otimes \varphi_1 \left(\begin{bmatrix} v_{1,1} & v_{1,2} \end{bmatrix}, \begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}\right)$$

$$=\left(\varphi_1\otimes\varphi_2+\operatorname{sign}(\sigma)\,\varphi_{\sigma(1)}\otimes\varphi_{\sigma(2)}\right)\left([v_{1,1}\ v_{1,2}],[v_{2,1}\ v_{2,2}]\right)$$

where $\sigma(1)=2$

$\sigma(2)=1$

Observe: If $\gamma \in S_n$,

then

$$\det\left(v_{\gamma(1)}, v_{\gamma(2)}, \ldots, v_{\gamma(n)}\right)$$

$$= \operatorname{Sign}(\gamma) \det\left(v_1, v_2, \ldots, v_n\right)$$

Definition: (alternating k-tensor)

Suppose $\gamma \in S_k$. Then

$T \in \mathcal{T}^k(\mathbb{R}^n)$ is called

alternating if

$$T(v_{\gamma(1)}, v_{\gamma(2)}, \ldots, v_{\gamma(k)})$$

$$= \operatorname{sign}(\gamma)\, T(v_1, v_2, \ldots, v_k)$$

$$\forall\ v_1, v_2, \ldots, v_k \in \mathbb{R}^n.$$

Notation: $\Lambda^k(\mathbb{R}^n)$

for the alternating k-tensors on $\mathbb{R}^n$. Can check that this is a subspace of $\mathcal{T}^k(\mathbb{R}^n)$.

Definition: (Alt)

Given $T \in \mathcal{T}^k(\mathbb{R}^n)$,

Define

$$\boxed{Alt(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, T(v_{\sigma(1)}, \dots, v_{\sigma(k)})}$$

Example 3: Let

$$\varphi_1, \varphi_2 \in \mathcal{T}^1(\mathbb{R}^2).$$

What is

$$\mathrm{Alt}(\varphi_1 \otimes \varphi_2)?$$

Check on basis elements:

$$\mathrm{Alt}(\varphi_1 \otimes \varphi_2)(e_i, e_j)$$

$$= \varphi_1(e_i)\varphi_2(e_j) - \varphi_1(e_j)\varphi_2(e_i)$$

$= 1 \quad \text{if} \quad i=1, j=2$

$= -1 \quad \text{if} \quad i=2, j=1$

$= 0 \quad \text{otherwise}.$